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Application of Reciprocal Substitution Method in Solving Some Improper Fractional Integrals

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Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional calculus, we use reciprocal substitution method to find two improper fractional integrals. Change of variables for fractional integral and a new multiplication of fractional analytic functions play important roles in this paper. In fact, our results are generalizations of traditional calculus results.**

Keywords: **Jumarie type of R-L fractional calculus, reciprocal substitution method, improper fractional integrals, change of variables for fractional integral, new multiplication, fractional analytic functions.**

I. INTRODUCTION

Fractional calculus is a branch of mathematical analysis, which studies several different possibilities of defining real order or complex order. In the second half of the 20th century, a large number of studies on fractional calculus were published in engineering literature. Fractional calculus is widely welcomed and concerned because of its applications in many fields such as mechanics, dynamics, control theory, physics, economics, viscoelasticity, electrical engineering, biology, and so on [1-11]

However, fractional calculus is different from ordinary calculus. The definition of fractional derivative is not unique. Common definitions include Riemann Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative and Jumarie's modification of R-L fractional derivative [12-16]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on the Jumarie's modified R-L fractional calculus, we use reciprocal substitution method to solve the following two improper α-fractional integrals:

$$\left({}_0I^{\alpha}_{+\infty}\right)\left[\left(\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right)^{\otimes_{\alpha}3}\right]^{\otimes_{\alpha}\left(-\frac{1}{2}\right)}\right)\right], \tag{1}$$

and

$$\left({}_0I^{\alpha}_{+\infty}\right)\left[\left(\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}2}+1\right]\otimes_{\alpha}\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}r}+1\right]\right)^{\otimes_{\alpha}(-1)}\right]. \tag{2}$$

Where $0<\alpha\leq 1$ and r is a real number. Change of variables for fractional integral and a new multiplication of fractional analytic functions play important roles in this paper. In fact, the results we obtained are generalizations of classical calculus results.

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II. PRELIMINARIES

At first, we introduce the fractional calculus used in this paper.

Definition 2.1 ([17])**:** If $0<\alpha\leq 1$, and x_0 is a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{3}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{4}$$

where $\Gamma(\ \ )$ is the gamma function.

In the following, some properties of Jumarie type of fractional derivative are introduced.

Proposition 2.2 ([18])**:** *If α,β,x_0,c are real numbers and $\beta\geq\alpha>0$, then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{5}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[c]=0. \tag{6}$$

In the following, we introduce the definition of fractional analytic function.

Definition 2.3 ([19])**:** Let x,x_0, and a_k be real numbers for all k, $x_0\in(a,b)$, and $0<\alpha\leq 1$. If the function $f_{\alpha}:[a,b]\to R$ can be expressed as an α-fractional power series, that is, $f_{\alpha}(x^{\alpha})=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_{\alpha}(x^{\alpha})$ is α-fractional analytic at x_0. In addition, if $f_{\alpha}:[a,b]\to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_{α} is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([20])**:** If $0<\alpha\leq 1$. Assume that $f_{\alpha}(x^{\alpha})$ and $g_{\alpha}(x^{\alpha})$ are two α-fractional power series at $x=x_0$,

$$f_{\alpha}(x^{\alpha})=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}, \tag{7}$$

$$g_{\alpha}(x^{\alpha})=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}. \tag{8}$$

Then

$$\begin{aligned}&f_{\alpha}(x^{\alpha})\otimes_{\alpha}g_{\alpha}(x^{\alpha})\\&=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}\otimes_{\alpha}\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}\\&=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(x-x_0)^{k\alpha}.\end{aligned} \tag{9}$$

Equivalently,

$$\begin{aligned}&f_{\alpha}(x^{\alpha})\otimes_{\alpha}g_{\alpha}(x^{\alpha})\\&=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_{\alpha}k}\otimes_{\alpha}\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_{\alpha}k}\\&=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_{\alpha}k}.\end{aligned} \tag{10}$$

Definition 2.5 ([21])**:** Suppose that $0 < \alpha \le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions. Then $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha n} = f_\alpha(x^\alpha)\otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the n-th power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha (-1)}$.

Definition 2.6 ([22])**:** Suppose that $0 < \alpha \le 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha k}. \tag{11}$$

Theorem 2.7 (change of variables for fractional integral) ([23])**:** *If $0 < \alpha \le 1$, g_α is α-fractional analytic at $x = a$, f_α is α-fractional analytic at $x = g_\alpha(a^\alpha)$ and the range of g_α contained in the domain of f_α, then $f_\alpha \circ g_\alpha$ is α-fractional analytic at $x = a$, and*

$$\left({}_{a}I_{b}^{\alpha}\right)\left[(f_\alpha \circ g_\alpha)(x^\alpha)\otimes_\alpha \left({}_{a}D_{x}^{\alpha}\right)[g_\alpha(x^\alpha)]\right] = \left({}_{g_\alpha(a^\alpha)}I_{g_\alpha(b^\alpha)}^{\alpha}\right)[f_\alpha(g_\alpha)], \tag{12}$$

for $a, b \in I$.

Definition 2.8: The smallest positive real number T_α such that $E_\alpha(iT_\alpha) = 1$, is called the period of $E_\alpha(ix^\alpha)$.

III. MAIN RESULTS

In this section, we will use reciprocal substitution method and change of variables for fractional integral to solve two improper fractional integrals.

Theorem 3.1: *If $0 < \alpha \le 1$, then the improper α-fractional integral*

$$\left({}_{0}I_{+\infty}^{\alpha}\right)\left[\left(\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha + 1\right)^{\otimes_\alpha 3}\right]^{\otimes_\alpha\left(-\frac{1}{2}\right)}\right)\right] = 2. \tag{13}$$

Proof Let $\frac{1}{\Gamma(\alpha+1)}x^\alpha = \left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes_\alpha(-1)}$. Using change of variables for fractional integral yields

$$\left({}_{0}I_{+\infty}^{\alpha}\right)\left[\left(\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha + 1\right)^{\otimes_\alpha 3}\right]^{\otimes_\alpha\left(-\frac{1}{2}\right)}\right)\right]$$

$$= \left({}_{0}I_{+\infty}^{\alpha}\right)\left[\left(\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha + 1\right)^{\otimes_\alpha 3}\right]^{\otimes_\alpha\left(-\frac{1}{2}\right)}\right)\otimes_\alpha \left({}_{0}D_{x}^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]\right]$$

$$= \left({}_{0}I_{+\infty}^{\alpha}\right)\left[\left(\left[\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes_\alpha(-1)} \otimes_\alpha \left(\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes_\alpha(-1)} + 1\right)^{\otimes_\alpha 3}\right]^{\otimes_\alpha\left(-\frac{1}{2}\right)}\right)\otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes_\alpha(-2)}\right]$$

$$= \left({}_{0}I_{+\infty}^{\alpha}\right)\left[\left(\left[\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes_\alpha(-4)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}t^\alpha + 1\right)^{\otimes_\alpha 3}\right]^{\otimes_\alpha\left(-\frac{1}{2}\right)}\right)\otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes_\alpha(-2)}\right]$$

$$= \left({}_{0}I_{+\infty}^{\alpha}\right)\left[\left(\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes_\alpha(2)} \otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}t^\alpha + 1\right)^{\otimes_\alpha\left(-\frac{3}{2}\right)}\right)\otimes_\alpha \left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes_\alpha(-2)}\right]$$

$$= \left({}_{0}I_{+\infty}^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha + 1\right)^{\otimes_\alpha\left(-\frac{3}{2}\right)}\right]$$

$$= \left[-2\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha + 1\right)^{\otimes_\alpha\left(-\frac{1}{2}\right)}\right]_0^{+\infty}$$

$$= 0 - (-2)$$

$$= 2\,.$$

Q.e.d.

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Theorem 3.2: *Let $0 < \alpha \leq 1$, and r be a real number, then the improper α-fractional integral*

$$\left({}_{0}I^{\alpha}_{+\infty}\right)\left[\left(\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha 2}+1\right]\otimes_\alpha\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha r}+1\right]\right)^{\otimes_\alpha(-1)}\right]=\frac{T_\alpha}{8}. \quad (14)$$

Proof Let $\frac{1}{\Gamma(\alpha+1)}x^{\alpha}=\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes_\alpha(-1)}$, and let

$$A=\left({}_{0}I^{\alpha}_{+\infty}\right)\left[\left(\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha 2}+1\right]\otimes_\alpha\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha r}+1\right]\right)^{\otimes_\alpha(-1)}\right]. \quad (15)$$

Then by change of variables for fractional integral, we obtain

$$\begin{aligned}
A&=\left({}_{0}I^{\alpha}_{+\infty}\right)\left[\left(\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha 2}+1\right]\otimes_\alpha\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha r}+1\right]\right)^{\otimes_\alpha(-1)}\right]\\
&=\left({}_{0}I^{\alpha}_{+\infty}\right)\left[\left(\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha 2}+1\right]\otimes_\alpha\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha r}+1\right]\right)^{\otimes_\alpha(-1)}\otimes_\alpha\left({}_{0}D^{\alpha}_{x}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]\right]\\
&=\left({}_{0}I^{\alpha}_{+\infty}\right)\left[\left(\left[\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes_\alpha(-2)}+1\right]\otimes_\alpha\left[\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes_\alpha(-r)}+1\right]\right)^{\otimes_\alpha(-1)}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes_\alpha(-2)}\right]\\
&=\left({}_{0}I^{\alpha}_{+\infty}\right)\left[\left(\left[\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes_\alpha 2}+1\right]\otimes_\alpha\left[\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes_\alpha r}+1\right]\right)^{\otimes_\alpha(-1)}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes_\alpha r}\right]. \quad (16)
\end{aligned}$$

Thus,

$$\begin{aligned}
2A&=A+A\\
&=\left({}_{0}I^{\alpha}_{+\infty}\right)\left[\left(\left[\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes_\alpha 2}+1\right]\otimes_\alpha\left[\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes_\alpha r}+1\right]\right)^{\otimes_\alpha(-1)}\otimes_\alpha\left[\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes_\alpha r}+1\right]\right]\\
&=\left({}_{0}I^{\alpha}_{+\infty}\right)\left[\left[\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes_\alpha 2}+1\right]\right]\\
&=\left[\arctan_\alpha(t^\alpha)\right]_0^{+\infty}\\
&=\frac{T_\alpha}{4}-0\\
&=\frac{T_\alpha}{4}. \quad (17)
\end{aligned}$$

Hence,

$$A=\left({}_{0}I^{\alpha}_{+\infty}\right)\left[\left(\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha 2}+1\right]\otimes_\alpha\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha r}+1\right]\right)^{\otimes_\alpha(-1)}\right]=\frac{T_\alpha}{8}. \quad \text{Q.e.d.}$$

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional calculus, we use reciprocal substitution method and change of variables for fractional integral to solve two improper fractional integrals. A new multiplication of fractional analytic functions plays an important role in this paper. In fact, our results are generalizations of ordinary calculus results. In the future, we will continue to use these methods to study the problems in fractional differential equations and applied mathematics.

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